Spec and Proj - Exercises

Ed Segal

1. A variety V is called *irreducible* if it can't be decomposed as $V = V_1 \cup V_2$ in a non-trivial way. Show that this is equivalent to (a) any two Zariski open subsets in V intersect, or (b) the ring $\mathbb{C}[V]$ is an integral domain.

2. (a) Show that Nullstellensatz fails over $\mathbb{R}$ by finding a polynomial f such that $I_{V(f)}$ is bigger than $\mathrm{rad}(f)$. Generalize to any non-algebraically-closed field.

   (b) Over $\mathbb{C}$ or $\mathbb{R}$ the ideal $I_{\mathbb{A}^n}$ is zero. What happens over $\mathbb{F}_p$?

3. Show that for any hypersurface $V(f) \subset \mathbb{A}^n$, the complement $\mathbb{A}^n \setminus V(f)$ is isomorphic to an affine variety.

4. Prove that every homomorphism $\mathbb{C}[W] \to \mathbb{C}[V]$ is induced from a regular map $V \to W$.

5. Let R be the ring $R = \mathbb{C}[x]/(x^2)$. Show that a homomorphism $\mathbb{C}[V] \to R$ is exactly the data of a point $p \in V$ and a vector which is tangent to V at p.

6. Let $V = V(y^2 - x^3) \subset \mathbb{A}^2$ and let $f : \mathbb{A}^1 \to V$ be the function:

   $$f : t \mapsto (t^2, t^3)$$

   Show that f is not an isomorphism but it is a bijection *(and even a homeomorphism in the usual complex topology).*

7. Compute the transition functions (*i.e.* the co-ordinate changes) between the standard affine charts in $\mathbb{P}^2$.

8. Find an explicit method to compactify any hypersurface in $\mathbb{A}^n$ to a hypersurface in $\mathbb{P}^n$.

9. (a) Let C be the projective curve $V(xy - z^2) \subset \mathbb{P}^2$. Construct a bijection $\mathbb{P}^1 \to C$.

   (b) What does this have to do with traceless 2×2 matrices of rank 1?

   (c) What happens if $xy - z^2$ is replaced with another quadratic form?

10. (a) Consider the complex affine curve $V_\epsilon = V(xy - \epsilon) \subset \mathbb{A}^2$ for $\epsilon \in \mathbb{C}$. Convince yourself that (i) for $\epsilon \neq 0$ the space V_ϵ is a cylinder, and (ii) as $\epsilon \to 0$ one circle in the cylinder collapses, leaving two discs glued at a point.

    (b) What's the topology of the complex projective plane curve $V(xy) \subset \mathbb{P}^2$? Now consider a small peturbation $f = xy + \epsilon g$ for some quadratic $g(x, y, z)$. What's the topology of $V(f)$? *Compare this with Q9.*

    (c) Now take a cubic plane curve of the form $V = V\big(xyz + \epsilon h(x, y, z)\big)$. Argue that V has the topology of a torus.