

Minimal surfaces

3 pts?

Geodesics $\rightarrow$ minimize length

in $\mathbb{R}^n \rightarrow$ lines, shortest distance between two points

Minimal surfaces $\rightarrow$ minimize area

in $\mathbb{R}^n \rightarrow$ soap films, smallest area or critical.



M .

$\vec{H}$ - mean curvature vector.

$H = (\partial_{e_1} e_1 + \partial_{e_2} e_2)^\perp$ $\leftarrow$ normal part.
example of a plane

Exercise: $M \xrightarrow{x} \mathbb{R}^n$, $X_t = X + tV \leftarrow$ vector field on M .

$$\left. \frac{d}{dt} \right|_{t=0} \text{Area}(M_t) = \left. \frac{d}{dt} \right|_{t=0} \int \sqrt{\det \langle (X+tV)_{e_i}, (X+tV)_{e_j} \rangle} =$$

$$= - \int_M \langle H, V \rangle \quad (\text{need to use divergence theorem})$$

&

$$\left. \frac{d}{dt} \right|_{t=0} \det(I + tA) = \text{tr}(A)$$

Laplace operator on M :

$$\Delta_M u = \nabla_{e_i} \nabla_{e_i} u - \nabla_{(\nabla_{e_i} e_i)} u.$$

Exercise $\Delta_M \vec{X} = \vec{H}$

Corollary: $M \subset \mathbb{R}^n$ - minimal iff
 $\Delta_M \vec{X} = 0$.

Max. principle $\Rightarrow$ all min. surfaces
are non-compact.

Show catenoid

Non-compact are generally harder.
There are two classical ways to
study compact minimal surfaces.

free boundary MS
in B^n

$$\begin{cases} \Delta \vec{X} = 0 \\ \partial_n \vec{X} = \vec{X} \end{cases}$$

MS in $S^n \subset \mathbb{R}^{n+1}$
 $\vec{H}|_{M \cap S^n} = 0$

$$\Delta \vec{X} = 2\vec{X}$$

Huxley 658

Introduction:

$$M \subset \mathbb{R}^3$$

X, Y - two vector fields on M

$$\partial_X Y = \nabla_X Y + \mathbb{I}(X, Y) \quad \leftarrow \text{second fundamental form.}$$

Exercise 0 $\mathbb{I}$ is a bilinear form.
 $\mathbb{I}(X, X) = |X|^2 \cdot \{\text{curvature of the curve in the direction of } X\}$.

Definition $\vec{H} = \text{tr } \mathbb{I} = \sum_{e_i} \mathbb{I}(e_i, e_i)$ - mean curvature vector. ($K = \det \mathbb{I}$)

What is the meaning of $\vec{H}$?

1st variation $M_t = M + tX$

$$\left. \frac{d}{dt} \right|_{t=0} \text{Area}(M_t) = - \int \langle \vec{H}, \vec{X} \rangle dV_g$$

Exercise 1 Prove it using

a) $\left. \frac{d}{dt} \right|_{t=0} \det(I + tA) = \text{tr}(A)$

b) divergence formula (consequence of the Stokes formula)

$$\int_M \text{div}(X) = \int_{\partial M} \langle \vec{X}, \vec{n} \rangle \quad \text{where}$$

$\uparrow$
outer normal

$$\text{div}(X) = \text{tr}(Y \mapsto \nabla_Y X) = \sum \langle \nabla_{e_i} X, e_i \rangle$$

Examples

① soap films (Plateau problem)

② • plane

• catenoid - surface of revolution for $y = \cosh(x)$ & its scalings.

Exercise 2 | plane & catenoid are the only minimal surfaces of revolution

Proposition There are no compact minimal surfaces in $\mathbb{R}^3$.

► $M \subset \mathbb{R}^n$ we can introduce Laplace - Beltrami operator (or Laplacian)

$$\Delta f = -\operatorname{div}(\nabla f) \quad (\text{in } \mathbb{R}^3, \Delta f = \sum (\partial_{x_i})^2 f)$$

Lemma $\Delta_M \vec{x} = \vec{H}$

► Exercise 3 $\Delta f = -\left(\sum \partial_{e_i} \partial_{e_i} f - \partial_{\nabla_{e_i}^M} f\right)$

$$\partial_{e_i} \vec{x} = e_i$$

$$-\Delta_M \vec{x} = \sum \partial_{e_i} e_i - \nabla_{e_i} e_i = \sum \Pi(e_i, e_i)$$

Maximum principle $\Rightarrow$ proposition.

Min surfaces can be defined for other ^{ambient} manifold, e.g. for $S^n \subset \mathbb{R}^{n+1}$

$$\nabla_x^{S^n} \gamma = \nabla_x^M \gamma + \mathbb{I}^{M \subset S^n}(x, \gamma)$$

Exercise 4 $\mathbb{I}^{S^n \subset \mathbb{R}^n}(x, \gamma) = \langle x, \gamma \rangle \vec{x}$

Corollary $M \subset S^n$ is minimal iff

$$\Delta_M \vec{x} = 2\vec{x}$$

$$\begin{aligned} \blacktriangleright H^{M \subset \mathbb{R}^{n+1}} &= \sum (\mathbb{I}^{M \subset S^n} + \mathbb{I}^{S^n \subset \mathbb{R}^n})(e_i, e_i) = \\ &= H^{M \subset S^n} + \underbrace{\sum \langle e_i, e_i \rangle \vec{x}}_{2\vec{x}} \quad \blacktriangle \end{aligned}$$

Exercise 5 $M \subset \mathbb{R}^{n+1}$ is such that $\Delta_M \vec{x} = 2\vec{x}$. Then $M \subset S^n$.

Common application of MS: relations between topology & curvature ($\sim$ geodesics)
Usually uses 2nd variation:

$$Q(\xi, \xi) = \int_M |\nabla^\perp \xi|^2 - \text{tr}(R_\Sigma(\cdot, \xi) \cdot, \xi) - |\langle \mathbb{I}, \xi \rangle|^2$$

$$Q(\varphi, \varphi) = \int_M |\nabla \varphi|^2 - (\text{Ric}_M(v, v) + |\mathbb{I}|^2) \varphi^2$$

for hypersurfaces.

Thm (Σ^n, g) - closed manifold, $\text{Ric}_n > 0$
 Then $H_{n-1}(\Sigma, g) = 0$.

► If $H^{n-1}(\Sigma, g) \neq 0 \Rightarrow$ (hard GMT result) $\exists$ minimal hypersurface (with sing. set of codim ≤ 8) minimizing volume in that homology class. Plug $\varphi=1 \Rightarrow$ contradiction (up to the sing. set). ◀

Geroch thm There is no metric on π^3 with $S_g > 0$



Exercise 6 For $M \subset \mathbb{R}^3$

$$(\text{Ric}(v, v) + |\text{II}|^2) = \frac{1}{2}(S + |\text{II}|^2) + K.$$

Claim: There exists a stable $\pi^2 \subset \pi^3$

$$Q(1, 1) = - \int_{\pi^2} \frac{1}{2}(S + |\text{II}|^2) + K \quad \& \quad \int K = 0 \quad \blacktriangleleft$$

Equator.

- A_{cc} - ! embedded annulus in B^3
- A_{cc} is the second smallest area.
- A_{cc} is the second smallest index

Any surface can be embedded as FMS in B^3

Clifford torus

- T_{cl} is the unique embedded torus in S^3 ✓
- T_{cl} is the second smallest area ✓
- T_{cl} is the second smallest index ✓

Any surface can be embedded as MS in S^3 ✓

Exercises

- ⑦ $S^2 \subset S^n$ has index $n-2$. Any other surface has index $\geq n+1$.

Hint: Consider variation fields given by the projections of constant vector fields to the normal bundle.

- ⑧ a) $M \subset S^3$, $M \approx S^2$.

Show that

$$\int_M 1 + |H|^2 \geq 4\pi. \quad (\text{easy})$$

- ⑧ b) for any $M \subset S^n$, $\int_M 1 + |H|^2 \geq 4\pi$. (hard)

Hints:

(i) $\int_M 1 + |H|^2$ is conformally invariant, i.e. if $\Phi: S^n \rightarrow S^n$ is a conformal automorphism, then

$$\int_{\Phi(M)} 1 + |\vec{H}_{\Phi(M)}|^2 = \int_M 1 + |\vec{H}_M|^2.$$

(ii) Consider conformal automorphisms given by: stereographic projection onto $\mathbb{R}^n$, then scaling of $\mathbb{R}^n$, then stereographic projection back.

(9) Let $M \subset S^3$ be minimal.

In local isothermic coordinates consider the expression

$$\omega_2 = (\partial_{\bar{z}} \vec{x}, \partial_{\bar{z}} \vec{x}) dz^2$$

$\omega_4 = (\partial_z^2 \vec{x}, \partial_z^2 \vec{x}) dz^4$, where $(\cdot, \cdot)$ is the standard $\mathbb{C}$ -bilinear form on $\mathbb{C}^3$.

(a) Show that $\omega_2 = 0$.

(b) Show that ω_4 is a holomorphic quartic differential.

(c) Suppose $M \approx S^2$, show that $\omega_4 = 0$.

(d) Use this to show that equatorial $\mathbb{S}^2 \subset \mathbb{S}^3$ is the unique minimal sphere in $\mathbb{S}^3$.