Lagrangian intersections

Yankı Lekili

LSGNT

March 27, 2025

Floer cohomology

Motivation & Heuristics

(M^{2n}, ω) symplectic manifold of dimension $2n$.
$L \subset M$ is a Lagrangian submanifold if $\omega|L = 0$ and $\dim L = n$.

- $L \subset T^*L$, $T^*_q \subset T^*L$ for any q. More generally, if $\alpha \in \Omega^1(L)$ is a closed 1-form, its graph $\{(q, \alpha_q) : q \in L\}$ is a Lagrangian submanifold.
- For b a regular value, $\mu^{-1}(b)$ is a Lagrangian in a toric variety with moment map $\mu : M^{2n} \to \mathbb{R}^n$. It is possible to consider generalizations of this to non-abelian group actions.
- $\mathbb{R}^n \subset \mathbb{C}^n$ is Lagrangian, more generally, the real part of a complex algebraic variety defined by real equations is isotropic and can be Lagrangian if it has the right dimension. $\mathbb{R}P^n \subset \mathbb{C}P^n$. The real solutions to $x_0^2 = x_1^2 + x_2^2 + \ldots x_n^2$ is a Lagrangian sphere in the complex quadric hypersurface in $\mathbb{C}P^n$ defined by this equation.

-Two major questions in symplectic topology:

- What are the topological restrictions on Lagrangian submanifolds in a given symplectic manifold M?
  Sample result: (Gromov) There is no closed Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ with $H^1(L) = 0$.
- Can two Lagrangians be displaced from each other by Hamiltonian isotopy? If not, what is the minimum number of interesections?
  Sample result: (Floer) Suppose that L is Hamiltonian isotopic to L' and $L \pitchfork L'$, $\omega_{|\pi_2(M,L)} = 0$ then

$$|L \cap L'| \geq \sum_i \mathrm{rk} H^i(L; \mathbb{Z}_2)$$

Recall, if $H : M \times [0,1] \to \mathbb{R}$ is a time-dependent Hamiltonian function, we can consider its flow ϕ_t which is the flow of the vector field defined by $\omega(X_{H_t}, \cdot) = dH_t$. Note that $\phi_0 = Id$. ϕ_t is called a Hamiltonian isotopy. Two Lagrangians L, L' are called Hamiltonian isotopic if there exists a Hamiltonian isotopy such that $L' = \phi_1(L)$.

Draw a picture of torus and discuss Floer's result

Floer cohomology

The proof of Floer's theorem is a consequence of the existence of Lagrangian Floer cohomology and its basic properties.

- Given two Lagrangians L, L' such that $\omega_{|\pi_2(M,L)} = \omega_{|\pi_2(M,L')} = 0$, there exists a cochain complex denoted by $CF^*(L, L')$ (over some field Λ) whose quasi-isomorphism type is an invariant of L and L' up to Hamiltonian isotopy.
- When the Lagrangians L and L' are transverse, this complex is generated by the intersection points in $L \cap L'$.
- When $L = L'$, there exists a quasi-isomorphism $CF^*(L, L) \simeq C^*(L)$.

Floer cohomology

Semi-infinite dimensional Morse theory

$$\Omega(L, L') = \{\gamma \in C^\infty([0,1], M) : \gamma(0) \in L, \gamma(1) \in L'\}$$

$$\hat{\omega}(\gamma)\xi = \int \omega(\xi, \dot{\gamma})dt$$

where ξ is a vector field along γ.
Since L, L' are Lagrangian, this is a closed 1-form on $\Omega(L, L')$ in the sense that if $u : S^1 \times [0,1] \to M$ is a contractible loop in $\Omega(L, L')$, then

$$\hat{\omega}(u) = \int u^*\omega = 0$$

Thus, we get a well-defined homomorphism $I_\omega : \pi_1(\Omega(L, L'), \gamma_0) \to \mathbb{R}$.

If I_ω vanishes, then $\hat{\omega} = dA_\omega$. In general, we have to work with the universal cover $\tilde{\Omega}(L, L')$ and get an "action functional"

$$A_\omega : \tilde{\Omega}(L, L') \to \mathbb{R}$$

given by

$$A_\omega(\tilde{\gamma}) = \int u^*\omega$$

here $u : [0,1] \times [0,1] \to M$ represents an element $\tilde{\gamma} \in \tilde{\Omega}$, i.e. $u(0,t) = \gamma_0(t)$ and $u(1,t) = \gamma(t)$.

To first approximation, Lagrangian Floer cohomology is the Morse cohomology of this action functional.

For simplicity, let us concentrate in the special case when I_ω vanishes. For example, this holds when L, L' are exact. This means, that $\omega = d\sigma$ and restriction of σ to L and L' are exact.

Let $\gamma_s(t)$ be a variation of $\gamma_0 = \gamma$ and write $u(s,t) = \gamma_s(t)$ and $\xi(t) = \frac{d}{ds}|_{s=0}\, \gamma_s(t)$.

$$A(\gamma_s) - A(\gamma_0) = \int_{t\in[0,1], s'\in[0,s]} u^*\omega$$

$$= \int_{t\in[0,1], s'\in[0,s]} \omega(du/ds', du/dt) ds' dt$$

$$= \int_0^1 \int_0^{s'} \omega(\frac{d\gamma_{s'}}{ds'}, \dot{\gamma}_{s'}) ds' dt$$

Critical points:

$$0 = \frac{\delta A}{\delta \gamma} = \frac{d}{ds}|_{s=0}\, A(\gamma_s) = \int_0^1 \omega(\xi, \dot{\gamma}) dt$$

for all ξ. Hence, we $\dot{\gamma} = 0$. In other words, γ is a constant path in $L \cap L'$.

Let J be a compatible complex structure, and define the metric $g(X, Y) = \omega(X, JY)$ on M. This gives an L^2 inner product on $\Omega(L, L')$ as follows:

$$\langle \xi, \eta \rangle = \int_0^1 g(\xi, \eta) dt = \int_0^1 \omega(\xi, J\eta)$$

$$\int_0^1 \omega(\xi, \dot{\gamma}) dt = dA_\gamma(\xi) = \langle \xi, \nabla A_\gamma \rangle = \int_0^1 \omega(\xi, J\nabla A_\gamma) dt$$

Hence, $\nabla A_\gamma = -J\dot{\gamma}$.

Gradient flow lines: $\frac{d}{ds}\gamma_s = \nabla A_{\gamma_s} = -J\dot{\gamma}$
If $u(s, t) = \gamma_s(t)$, then we get the **J-holomorphic curve equation**

$$\frac{\partial u}{\partial s} + J\frac{\partial u}{\partial t} = 0$$

Issues with Morse theory in ∞-dimensions

- We want to study gradient flow of A on $\Omega(L, L')$ but $\nabla A_\gamma = -J\dot{\gamma}$ is not even tangent to $\Omega(L, L')$ because $-J\dot{\gamma}(0)$ and $-J\dot{\gamma}(1)$ need not be tangent to L and L' resp. (The problem is that the L^2 inner product on $\Omega(L, L')$ is not complete. The gradient of a functional can be defined via Riesz representation theorem which requires a complete metric. Indeed $-J\dot{\gamma}$ lies in the L^2-completion of $T_\gamma(\Omega(L, L'))$.) Floer's solution is to directly study the J-holomorphic curve equation as an elliptic PDE in finite-dimensions, rather than an ODE in infinite dimensions. (non-equivalent!).
- $\mathrm{Hess} A(\xi, \eta) = \langle \xi, D\eta \rangle$ where $D = -J\frac{d}{dt}$ is the 1-d Dirac operator. This has infinitely many positive and negative eigenvalues. Hence, the Morse index is infinite.
  Floer's solution is to define a relative index depending on a path between the critical points which is finite-dimensional can be computed topologically via the Atiyah-Singer index theory.

Floer cohomology

For any field $\mathbb{K}$ (which we take $\mathbb{Z}_2$ for simplicity) define the field extension

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{r_i} | a_i \in \mathbb{K}, r_i \in \mathbb{R}, \lim_{i\to\infty} r_i = +\infty\}$$

This is called the Novikov field.
Suppose L, L' transverse Lagrangians (compact, oriented, spin).
Define the vector space

$$CF(L, L') = \bigoplus_{p \in L \cap L'} \Lambda \cdot p$$

We define a differential

$$\partial p = \sum_{q \in L \cap L', \beta \in \pi_2(M, L, L'), \mu(\beta)=1} \#\mathcal{M}(p, q, \beta, J) T^{\omega(\beta)} q$$

$\mathcal{M}(p, q, \beta, J)$ is the moduli space of strips that we discuss next.

$\widehat{\mathcal{M}}(p, q, \beta, J)$ is the set of J-holomorphic strips

$$u : \mathbb{R} \times [0, 1] \to M$$

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0,$$

$$u(s, 0) \in L, \quad u(s, 1) \in L'$$

$$\lim_{s\to+\infty} u(s, t) = p, \quad \lim_{s\to-\infty} u(s, t) = q$$

$$E(u) = \int u^*\omega = \int\int \left|\frac{\partial u}{\partial s}\right|^2 dsdt < \infty$$

$\widehat{\mathcal{M}}(p, q, \beta, J)$ is the set of u satisfying the above conditions with $[u] = \beta \in \pi_2(M, L, L')$. Finally,

$$\mathcal{M}(p, q, \beta, J) = \widehat{\mathcal{M}}(p, q, \beta, J)/\mathbb{R}$$

where $\mathbb{R}$ acts by $r \cdot u(s, t) = u(s + r, t)$.

Here are some basic properies of $HF(L, L')$.

- $HF(L, L')$ does not depend on the choice of J
- $HF(\phi(L), L') \simeq HF(L, L')$ for an Hamiltonian diffeomorphism ϕ.
- $HF(L, L') = 0$ if L and L' are disjoint. (So Floer cohomology is an obstruction Hamiltonian displaceability.)
- $\chi(HF) = \chi(CF) = (-1)^{n(n+1)/2}[L] \cdot [L']$ (if L, L' are oriented).
- (Poincaré Duality) $HF^*(L, L') \simeq HF^{n-*}(L', L)^\vee$

Example (draw a picture)

Take $M = T^*S^1 (\simeq \mathbb{C}^*)$., L zero section, L' as drawn.

$$CF(L, L') = \Lambda \cdot p \oplus \Lambda \cdot q$$

$$\partial p = (T^{\omega(u)} - T^{\omega(v)})q$$

So, if $\omega(u) = \omega(v)$, then $\partial = 0$, and $HF(L, L') = \Lambda^{\oplus 2} = H^*(S^1; \Lambda)$.

If $\omega(u) \neq \omega(v)$, $HF(L, L') = 0$.

Note that if $\omega(u) \neq \omega(v)$ we can find a Hamiltonian isotopy ϕ_H such that $L \cap \phi_H(L') = \emptyset$.

If $\omega(u) = \omega(v)$ then L and L' are Hamiltonian isotopic.

$\mathcal{M}(p, q, \beta, J)$

In general, there are serious difficulties in making the set $\mathcal{M}(p, q, \beta, J)$ a "moduli space" i.e. equip it with some kind of manifold-like structure (and compactify) so that we can "count" the number of points in a meaningful way that does not depend on the choices.

There are three main technical steps in making this count rigorous.

- Transversality
- Compactness
- Gluing

Transversality

Let $\mathcal{B}$ be Banach manifold of maps $u : \mathbb{R} \times [0,1] \to M$, $[u] = \beta$ with boundary conditions and asymptotic conditions as before. Let $\mathcal{E}$ Banach vector bundle over $\mathcal{B}$ with fiber over u given by $\Omega^{0,1}(\mathbb{R} \times [0,1], u^* TM)$

$$\overline{\partial}_J : \mathcal{B} \to \mathcal{E}$$

is a section just that

$$\mathcal{M}(p, q, \beta, J) = \overline{\partial}_J^{-1}(0)$$

$\overline{\partial}_J$ is Fredholm i.e its linearization $D = D_{\overline{\partial}_J}$ is a Fredholm operator. Its index

$$\mathrm{Ind}(D_{\overline{\partial}_J}) = \dim kerD - \dim cokerD = \mu(\beta)$$

is the expected dimension of the moduli space (Maslov index). For a generic (domain dependent $J = J_t$), the operator D is surjective i.e. $\overline{\partial}_J$ is transverse to the zero section. Thus, we get a manifold of dimension $\mu(\beta)$.

Gromov compactness

Consider a sequence J_n of almost complex structures on M which converge uniformly to J_∞, and (C_n, j_n) a sequence of Riemann surface with boundary of fixed topological type (that means, there exist parametrizations $\delta_n : (\Sigma, \partial\Sigma) \to (C_n, j_n)$ from a fixed real surface Σ).

Let $u_n : C_n \to M$ a sequence of (j_n, J_n) holomorphic curves.

Theorem. Suppose that the energy $E(u_n) < \infty$ is bounded uniformly. Then there exists a subsequence which converges, up to reparametrization, to a **stable map**, that is a nodal **tree** (C_∞, j_∞) of (j_∞, J_∞) holomorphic curves.

Draw a picture of a stable map.

Besides possible degeneration of the domain to a nodal curve, the main issue is **bubbling** of spheres.
Let $u_n : \mathbb{C}P^1 \to \mathbb{C}P^1 \times \mathbb{C}P^1$ given by

$$(x : y) \to (x : y), (ny : x)$$

In affine chart $t = y/x$, we have $t \to (t, \frac{1}{nt})$. Then away from the origin, we have uniform convergence to $t \to (t, 0)$. So, the limit curve seems to be one axis (missing the other axis).
But, if we reparametrize $s = nt$, then get $s \to (\frac{s}{n}, \frac{1}{s})$. Hence, we get uniform convergence away from $t = \infty$ to $s \to (0, \frac{1}{s})$ (the other axis).

The main point is that the bubbling regions are where $\sup|du_n| \to \infty$. In these regions, we rescale the domain $v_n(z) = u_n(z_n^0 + \epsilon_n z)$ for $\epsilon_n \to 0$ suitably chosen. A subsequence of v_n converges to a map $v_\infty : \mathbb{C} \to X$ which completes to a map from $\mathbb{C}P^1$ by removable singularity theorem. This latter map is called a "bubble".

There is a version of Gromov compactness for curves with boundary on Lagrangians. We now also get **disk bubbles**. Example: Let L_0 be real axis, and L_1 the unit circle. View $\mathbb{R} \times [0, 1]$ via the conformally equivalent $\{z \in \mathbb{C} : \mathrm{Im}(z) > 0, |z| < 1\}$, we can consider the map

$$u_a(z) = \frac{z^2 + a}{1 + az^2}$$

for some $a \in (-1, 1)$.

Draw the two ends of this moduli space. $a \to -1$ broken strip, $a \to 1$ constant strip and a disk bubble.

Proof of $d^2 = 0$

Let's first recall how this goes in Morse (co)homology in the finite-dimensional setting.

In Floer theory, the proof is similar to that for Morse cohomology.

Fix p, q and look and 1-dimensional moduli space of (index 2) strips $\mathcal{M}(p, q; \beta, J)$ between p, q. By Gromov compactness this can be compactified to $\overline{\mathcal{M}}(p, q; \beta, J)$ and boundary terms are one of the following:

- Broken strips connecting p to q
- Configuration with interior sphere bubble.
- Configuration with boundary disk bubble.

Gluing

By topological assumption, there are no disc or sphere bubbles. The gluing theorem says that **every** broken strip is locally the limit of a unique family of index 2 strips.

Here is the statement of the gluing result

$$\partial\overline{\mathcal{M}}(p,q;\beta,J) = \bigsqcup_{\substack{r\in L\cap L' \\ \beta'+\beta''=\beta \\ \mu(\beta')=\mu(\beta'')=1}} \left(\mathcal{M}(p,q;\beta',J)\times\mathcal{M}(q,r;\beta'',J)\right)$$

Now $d^2=0$ follows as before by the miraculously trivial fact the number of points in the boundary of a 1-manifold is always even!

self Floer cohomology

For $HF(L, L)$, one way to define it is via considering $L' = \phi_H(L)$ for some Hamiltonian H, and use $HF(L, \phi_H(L))$.

There is also an alternative "Morse-Bott" model which often is easier to compute. One takes $CF^*(L, L) = C^*(L)$ singular cohomology complex, and define

$$\partial_{Floer} = \partial_{sing} + \sum_{\beta \neq 0, \beta \in \pi_2(M,L)} ev_{-1*}(\overline{\mathcal{M}}_{0,2}(\beta) \cap ev_1^*(C))\, T^{\omega(\beta)}$$

$\overline{\mathcal{M}}_{0,2}$ is moduli space of holomorphic disks with boundary on L and two marked points on its boundary. (draw a picture!).

Product structures

Let L_0, L_1, L_2 be three Lagrangians which intersect tranversely and do not bound any holomorphic disks. We have a product opereation

$$CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2)$$

Higher products

More generally, given $L_0, L_1, \dots L_n$, we can define a higher product

$$\mathfrak{m}_n : CF(L_{n-1}, L_n) \otimes CF(L_{n-2}, L_{n-1}) \otimes \dots CF(L_0, L_1) \to CF(L_0, L_n)$$

by counting holomorphic $(n+1)$-gons. (For $n > 2$, the domain has moduli, so it is important to understand this moduli space which can be parametrized by Stasheff-Tamari polytopes).

All of these together satisfy the A_∞-equations

$$\sum_{m,n} \pm \mathfrak{m}_{d-m+1}(a_d, \dots, a_{n+m+1}, \mathfrak{m}_m(a_{n+m}, \dots, a_{n+1}), a_n, \dots a_1) = 0.$$

Picture proof of A_∞ relations

Fukaya categories

Given a symplectic manifold M, one defines an A_∞ category whose objects are Lagrangian submanifolds, morphisms are given by Floer complexes and the product operations are defined by J-holomorphic polygon counts. One then considers a formal triangulated enveloppe of this (complexes of Lagrangians) and calls that the Fukaya category of M, $\mathcal{F}(M)$. There are various flavours of Fukaya categories associated to different setting such as compact Fukaya category, wrapped Fukaya category, Fukaya-Seidel category..etc. All of these involve Lagrangian submanifolds and counts of holomorphic polygons between them. Besides being a powerful invariant of the underlying symplectic structure on M, $\mathcal{F}(M)$ appears prominently in the homological mirror symmetry conjecture.

If there is more time ...

make an impromptu on mirror symmetry

References

- D. Auroux, A beginner's introduction to Fukaya categories
- P. Seidel, Fukaya categories and Picard-Lefschetz theory
- I. Smith, A symplectic prolegomenon
- S. Ivashkovic, V. Shevchisin Complex curves in almost complex manifolds and meromorphic hulls.
- Kenji Fukaya, Yong-Geun Oh, Hiroshi Ohta, and Kaoru Ono. Lagrangian intersection Floer theory: anomaly and obstruction.
- A. Floer, Morse theory for Lagrangian intersections.
- A. Floer, Witten's complex and infinite-dimensional Morse theory
- E. Witten, Supersymmetry and Morse theory