**LSGNT Topics in Geometry: Exercises for Mapping Class Groups**

For the following exercises, let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus.

**1.** The purpose of this exercise is to prove that $\mathrm{MCG}(\mathbb{T}^2)$ is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$, and to find an explicit generating set for it. Here we consider $\mathrm{MCG}(\mathbb{T}^2)$ as the group of orientation-preserving homeomorphisms of $\mathbb{T}^2$ up to *homotopy.*

i) Show that two orientation-preserving homeomorphisms $f$ and $g$ of the torus are homotopic to each other if and only if they have the same induced map on $\pi_1(\mathbb{T}^2)$. (Hint: Use the fact that $\mathbb{T}^2$ is a $K(\pi,1)$.)

ii) Use the isomorphism $\pi_1(\mathbb{T}^2) \cong H_1(\mathbb{T}^2;\mathbb{Z})$ to deduce that $f$ and $g$ are homotopic to each other if and only if they have the same induced map on $H_1(\mathbb{T}^2;\mathbb{Z})$.

iii) Note that the action of an orientation-preserving homeomorphism $f$ of $\mathbb{T}^2$ preserves the intersection pairing on $H_1(\mathbb{T};\mathbb{Z})$, and so represents as element of $\mathrm{Sp}(2,\mathbb{Z}) \cong \mathrm{SL}(2,\mathbb{Z})$. Deduce that there is a well-defined homomorphism $\Phi\colon \mathrm{MCG}(\mathbb{T}^2) \to \mathrm{SL}(2,\mathbb{Z})$ that is injective.

iv) Show that the image of the Dehn twists about the standard curves $\alpha = (1,0)$ and $\beta = (0,1)$ are given by the matrices

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Prove that the above matrices generate $\mathrm{SL}(2,\mathbb{Z})$, and conclude that $\Phi$ is surjective.

v) Conclude that $\Phi$ is an isomorphism, and that $\mathrm{MCG}(\mathbb{T}^2)$ is generated by the Dehn twists about $\alpha$ and $\beta$.

**2.** Let $S$ be a compact connected orientable surface. Two simple closed curves $\alpha$ and $\beta$ on $S$ are *of the same type* if there is an orientation-preserving homeomorphism $\phi\colon S \to S$ such that $\phi(\alpha) = \beta$.

i) A simple closed curve $\alpha$ is called *nonseparating* if the surface $S \backslash \alpha$ is connected. Show that any two nonseparating simple closed curves on $S$ are of the same type.

ii) For $S$ a surface of genus 1, 2, and 3, draw representatives for all types of simple closed curves on $S$.

iii) Show that there are only finitely many types of simple closed curves on $S$.

**3.** Let $S$ be a closed orientable surface of genus 2, and $\iota\colon S \to S$ be the hyperelliptic involution.

i) Show that $\iota$ is in the center of $\mathrm{MCG}(S)$, by exhibiting an explicit set of Dehn twists that generate $\mathrm{MCG}(S)$ and such that each of them commutes with $\iota$.

ii) During the lecture we showed that any element $f$ in the center of $\mathrm{MCG}(S)$ fixes the isotopy class of every simple closed curve. Deduce that for every simple closed curve $\alpha$ on $S$, the curve $\iota(\alpha)$ is isotopic to $\alpha$.

**4.** Let $S$ be a compact orientable surface of genus $g \geq 3$. During the lecture, we used Dehn twists to show that the center of the mapping class group $\mathrm{MCG}(S)$ of $S$ is trivial.

i) Let $\alpha$ be a simple closed curve on $S$, and $f\colon S \to S$ be a mapping class. Denote the Dehn twist about $\alpha$ by $T_\alpha$. Show that for every integer $n$

$$(T_{f(\alpha)})^n = f \circ (T_\alpha)^n \circ f^{-1}.$$

ii) Let $H < \mathrm{MCG}(S)$ be a finite index subgroup. Imitate the proof given in the lecture to show that the center of $H$ is trivial.

**5.** Let $S$ be a closed orientable surface of genus $g \geq 2$. Use the steps below to show that if $f\colon S \to S$ is a periodic orientation-preserving diffeomorphism of order $n \geq 2$, then the action of $f$ on $H_1(S;\mathbb{Z})$ is not identity. In other words, the Torelli subgroup of the mapping class group contains no torsion element (This is also true for $g = 1$ as a consequence of the isomorphism in Question 2).

i) Show that for any finite group $H$ acting on any smooth manifold $S$, there is a Riemannian metric $g$ on $S$ that is invariant under the action of $H$, meaning that for any $h \in H$ we have $h^*g = g$ (Hint: use the following averaging technique: if $\mathrm{id} = h_1, h_2, \cdots, h_n$ are the elements of $H$, and $g$ is any Riemannian metric then show that the metric $h_1^*g + \cdots + h_n^*g$ is invariant under $H$). Conclude that there is a Riemannian metric $g$ on $S$ that is invariant under the action of the finite cyclic group generated by $f$.

ii) Assume that the action of $f$ on $H_1(S;\mathbb{Z})$ is the identity. Use the Lefschetz fixed point theorem to deduce that $f$ has at least one fixed point with negative index.

iii) A Riemannian isometry $f\colon S \to S$ is determined by the image $q = f(p)$ of a point $p$ and the derivative $Df_p\colon T_pS \to T_qS$. Let $p$ be an arbitrary fixed point of $f$ (such points exist by the previous part). Use the fact that $f$ is orientation-preserving to deduce that $Df_p$ is a rotation and hence $f$ has index 1 at $p$.

**6.** Let $S$ be a compact orientable surface. The Nielsen–Thurston classification states that each element of the mapping class group $\mathrm{MCG}(S)$ has a representative that is of a particular form. The purpose of this exercise is to prove the Nielsen–Thurston classification for the special case $\mathrm{MCG}(\mathbb{T}^2) \cong \mathrm{SL}(2,\mathbb{Z})$. Let $A$ be an element of $\mathrm{SL}(2,\mathbb{Z})$.

i) Show that the characteristic polynomial of $A$ has the form $x^2 - \mathrm{tr}(A)\,x + 1 = 0$, where $\mathrm{tr}(A) \in \mathbb{Z}$ is the trace of the matrix $A$.

ii) Show that if $|\mathrm{tr}(A)| < 2$, then $A^{12} = id$.

iii) Show that if $|\mathrm{tr}(A)| = 2$, then $A$ has 1 or $-1$ as eigenvalue. Use this to show that there is an essential simple closed curve on $\mathbb{T}^2$ that is invariant under $A$.

iv) Show that if $|\mathrm{tr}(A)| > 2$, then the eigenvectors of $A$ have irrational slopes. Deduce that $A$ does not fix the isotopy class of any essential simple closed curve on $\mathbb{T}^2$. In this case, $A$ is called *Anosov*.

Look up, say on Wikipedia, the statement of the Nielsen–Thurston classification for surfaces of genus $g \geq 2$. The proof in higher genera is more complicated.