Exercises

LSGNT Topics
Calculus of Variations

1. Let $u \in W^{1,2}(B)$ solve $\int \nabla u \nabla \phi = 0$ for all $\phi \in C_c^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is an open set. Prove that u is C^2 in Ω following these steps.

   (a) Assume that $f \in C^3(\Omega)$ solves $\Delta f = 0$ in $\Omega \subset \mathbb{R}^n$ (open set). Prove that there exists a constant $c(n)$ such that for $B_r(x) \subset \Omega$

   $$\sup_{B_{R/2}(x)} |\nabla f| \leq \frac{c(n)}{R} \sup_{B_R(x)} |f|.$$

   Hints: consider $D_k f$ (any partial derivative), note that $\Delta(D_k f) = 0$, use the mean value property for harmonic functions and the divergence theorem with vector field $f\, e_k$ (where e_k is the k-th vector of the standard basis of $\mathbb{R}^n$.

   (b) Mollify u with a standard mollifier ρ_σ, let $u_\sigma = u \star \rho_\sigma$, in an open set $\Omega' \subset\subset \Omega$. Check that $\Delta u_\sigma = 0$ in Ω'.

   (c) Prove that u_σ converges to u in $C^2_{\text{loc}}(\Omega)$.
   Hints: Use (a) and iterated versions of it to obtain uniform bounds (independent of σ) for $|D^3 u_\sigma|$ in a suitable interior set and use said bounds to show that u_σ converges to a limit in C^2. You may use the mean value theorem to control the supremum of a harmonic function by its L^1 norm, and Ascoli–Arzelà's theorem to extract a limit in C^2.