DYNAMICAL SYSTEMS

NATTALIE TAMAM

1. Definition and main questions

Most generally, a dynamical system is a tuple (T, X, φ), where X is a set (the space), T is a monoid (the time), and for every $t \in T$ is $\varphi(t)$ an action on this set, i.e., a function $\varphi(t) : X \to X$ so that

$$\varphi(e)(x) = x,$$
$$\varphi(t_1 + t_2)(x) = \varphi(t_2)(\varphi(t_1)(x)).$$

There are different types of dynamical systems, for example:

- **Discrete.** X is discrete, for example a graph, or T is discrete, for example $T = \mathbb{Z}$.
- **Geometrical.** X is a manifold, i.e. locally a Banach space or Euclidean space.
- **Measurable.** X is a measurable space and $\varphi(t)$ is measure preserving. That is, (X, Σ, μ), where Σ is a σ-algebra on X and μ is a (finite) measure on (X, Σ), and for every $s \in \Sigma$ we have $\varphi(t)^{-1}(s) \in \Sigma$ and $\mu(\varphi(t)^{-1}(s)) = \mu(s)$.

The type of questions one can ask are:

- What are the orbit closures? (Ergodicity)
- Are the periodic orbits?
- Are there dense orbits?
- What is a common orbit?
- Classification of $\varphi(T)$-invariant measures.

2. Ergodic theory and some examples

Let us recall:

Definition 2.1. *A set $\Sigma \subseteq 2^X$ is a Σ-**algebra** if*

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- $\varnothing \in \Sigma$,
- $s_1, s_2 \in \Sigma$ *implies* $s_1 \cap s_2 \in \Sigma$,
- $s \in \Sigma$ *implies* $X \backslash s \in \Sigma$,
- $s_1, s_2, \dots \in \Sigma$ *implies that* $\bigcap_{i=1^\infty} s_i \in \Sigma$.

Definition 2.2. *A measure* μ *is a map* $\mu : \Sigma \to \mathbb{R}_+ \cup \{\infty\}$, *such that*

- $\mu(\varnothing) = 0$,
- $\mu\left(\bigcup_{i=1}^{\infty} s_i\right) = \sum_{i=1}^{\infty} \mu(s_i)$ *for every pairwise disjoint sequence of sets* $\{s_i\}_{i \in \mathbb{N}}$.

Example 2.3. *On* $[0,1]$ *we can define the* σ*-algebra generated by all open intervals* (a,b), $a \leqslant b \in \mathbb{R}$ *and the measure* $\mu([a,b]) = b - a$. *This is the natural measure on intervals, called the* ***Lebesgue measure****.*

Now, we have $X = [0,1)$ *as a measurable set with the above* σ*-algebra and* μ *the Lebesgue measure. Let us define an action,* $T = T_a : X \to X$ *by*

$$T_a(x) = x + a \bmod 1.$$

Then T *is measure preserving. Fix* $a \in \mathbb{R} \backslash \mathbb{Q}$. *Then, for any* $x \in [0,1)$ *the orbit*

$$\{T^n(x) : n \in \mathbb{Z}\}$$

is infinite. By the axiom of choice one can construct a set $s \subset [0,1)$ *which contains exactly one element of every orbit of* T.

Let $s_n = T^n(s)$. *Then, we have the following:*

- *For any* $n \neq m$, $s_n \cap s_m = \varnothing$.
- $\bigcup_{n \in \mathbb{Z}} s_n = [0,1)$.

If the set A *was Lebesgue measurable, then*

$$\begin{aligned} 1 = \mu([0,1)) &= \sum_{n \in \mathbb{Z}} \mu(s_n) \\ &= \sum_{n \in \mathbb{Z}} \mu(s). \end{aligned}$$

Since $\mu(s) = 0$ *or* $\mu(s) > 0$ *and in both cases we get a contradiction,* s *is non-measurable.*

Assume (X, Σ, μ, T) is a measurable dynamical system. In particular, T is measure preserving.

Definition 2.4. *Let* $T : X \to X$ *be a measure-preserving transformation on a measure space* (X, Σ, μ), *with* $\mu(X) = 1$. *Then* T *is* ***ergodic*** *if for every* $s \in \Sigma$ *we have*

$$T^{-1}(s) = s \Rightarrow \mu(s) \in \{0,1\}.$$

Example 2.5. *Given the previous example - $X = [0,1)$ with μ the Lebesgue measure (and the appropriate σ-algebra), and $T_a : X \to X$ is the rotation*

$$T_a(x) = x + a \bmod 1.$$

This map is ergodic with respect to the Lebesgue measure when a is irrational and is not when a is rational.

Example 2.6. *Fix $0 \leqslant p \leqslant 1$, and let us consider the infinite coin toss with probability p. That is, $X = \{0,1\}^{\mathbb{Z}}$ (one may also consider $X = \{0,1\}^{\mathbb{N}}$), and every element in X is an infinite string of integers in $\{0,1\}^{\mathbb{N}}$. If we give this set the product topology, we can consider the smallest σ-algebra Σ which contains all the open sets.*

*A **cylinder** is a set of the form*

$$A = \{x \in X : x_i = a_i \text{ for } i \in I\}, \tag{2.1}$$

for some finite $I \subset \mathbb{Z}$ and fixed $a_i \in \{0,1\}$ for all $i \in I$. Σ is the σ-algebra generated by all cylinders. The measure μ on X is defined by its definition on cylinder sets: $\mu(A) = \prod_{i \in I} p_i$, where

$$p_i = \begin{cases} 1-p & \text{if } a_i = 0, \\ p & \text{if } a_i = 1 \end{cases}$$

Consider the transformation $T : X \to X$ that shifts every element left, so that

$$T(x)_i = x_{i-1}.$$

*It preserves the measure of all cylinder sets, which generate Σ so it is measure-preserving. T is called a **Bernoulli Shift**.*

Theorem 2.7 (Birkhoff–Khinchin theorem - ergodic case)**.** *Let f be measurable (i.e., the preimage of any measurable set is measurable) such that $\int_X |f| d\mu < \infty$ and assume T is ergodic. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) = \int_X f d\mu.$$

Dynamical Systems Problem Set

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(1) In Example 2.3.
   (a) Show that T is measure preserving.
   (b) Prove the two properties of the sequence of sets $\{s_n\}_{n\in\mathbb{Z}}$.

(2) Show that the dynamical system $X = [0,1)$, μ is the Lebesgue measure, Σ is the σ-algebra of the Lebesgue measure, and $T : X \to X$ is the rotation
$$T(x) = x + 1/2 \bmod 1,$$
is not ergodic.

(3) Show that the Bernoulli Shift (from Example 2.6) is ergodic.

(4) Let $([0,1], T)$ be the dynamical system on the space $[0,1]$ with a transformation $T : X \to X$ defined by
$$T(x) = \begin{cases} 2x & \text{if } 0 \leqslant x \leqslant 1/2, \\ 2x - 1 & \text{if } 1/2 < x. \end{cases}$$
   (a) Show that T preserves the Lebesgue measure.
   (b) Can you find a periodic orbit of the system, i.e., a point $x \in [0,1]$ so that $\{T^n(x) : n \in \mathbb{N}\}$ is a finite set?
   (c) Can you find a non-periodic orbit of the system, i.e., a point $x \in [0,1]$ so that $\{T^n(x) : n \in \mathbb{N}\}$ is infinite?
   (d) Can you classify the periodic orbits?