LSGNT Blow ups exercises

October 2024

1. Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal and let $I = (f_1, \dots, f_r)$ and $I = (f'_1, \dots, f'_s)$ be two sets of generators for I. Using the explicit description of the blow up in terms of generators, let $X \subset \mathbb{A}^n \times \mathbb{P}^{r-1}$ be the blow up using $f_1, \dots, f_r$ and let $X' \subset \mathbb{A}^n \times \mathbb{P}^{s-1}$ be the blow up using $f'_1, \dots, f'_s$. Show that $X \cong X'$.

2. Let $L \subset \mathbb{A}^n$ be a linear subspace. What is the exceptional divisor of blow up at L?

3. Compute the resolution of $\{x^p + y^q = 0\} \subset \mathbb{A}^2_{\mathbb{C}}$ using blow ups. How many blow ups does it take?

4. Compute the blow up of the Whitney umbrella $\{x^2 + y^2 z = 0\} \subset \mathbb{A}^3_{\mathbb{C}}$ at the origin in each of the three natural coordinate charts.

5. Blow up $\mathrm{Spec}(\mathbb{Z}[x])$ at the point (p, x). What is the exceptional divisor?

6. Let $a_1, \dots, a_n$ be positive integers. We define the *weighted blow up* of $\mathbb{A}^n$ at $I = (x_1, \dots, x_n)$ to be $\mathbf{Proj}_{\mathbb{A}_n} R(I)$ where we stipulate that x_i has graded weight $= a_i$. Show that $\{x^p + y^q = 0\}$ can be resolved by a single weighted blow up.

7. Show that the blow up of $\mathbb{P}^2$ in two points is isomorphic to the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point. Try to generalise this statement by replacing $\mathbb{P}^2$ and $\mathbb{P}^1 \times \mathbb{P}^1$ by arbitrary Hirzebruch surfaces.

8. We define the *Cremona involution* to be the rational map $C \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ given by $C([x : y : z]) = [1/x : 1/y : 1/z]$. Show that C is a birational map. Where is C not well defined? (Hint: its only at finitely many points).

9. Show that there exists a diagram

$$\begin{array}{ccccc} & & W & & \\ & \swarrow_{g} & & \searrow^{f} & \\ C\colon \mathbb{P}^2 & & \dashrightarrow & & \mathbb{P}^2 \end{array}$$

   where f and g are each a blow of $\mathbb{P}^2$ at 3 points.

10. Let $S^3 \subset \mathbb{C}^2$ be the unit sphere about the origin. Let $X \to \mathbb{C}^2$ be the blow up at the origin with exceptional divisor E. Show that there is a morphism $X \to E$. Show that the restricted map (of topological spaces) $S^3 \to E$ is the Hopf fibration.

11. The surface singularity $\{xy + z^n = 0\} \subset \mathbb{A}^3$ is a type of Du Val singularity called an A_n-singularity. Resolve this singularity by blow ups. What is the exceptional locus of this resolution? What is intersection pairing of the exceptional locus?

12. The other types of Du Val singularities are D_n-singularities given by $\{x^2 + zy^2 + y^{n-1} = 0\}$ where $n \geq 4$, E_6-singularities given by $\{x^2 + y^3 + z^4 = 0\}$, E_7-singularities given by $\{x^2 + y^3 + yz^3 = 0\}$ and E_8-singularities given by $\{x^2 + y^3 + z^5 = 0\}$. Compute the resolutions of these singularities.

13. Let X be a smooth surface and let $\sum E_i \subset X$ be a simple normal crossings divisor. We define the *dual graph* $D(\sum E_i)$ as follows. The vertices of $D(\sum E_i)$ correspond to the irreducible components of $\sum E_i$ and there is an edge between v_{E_i} and v_{E_j} for each point of $E_i \cap E_j$. Compute the dual graphs of the exceptional divisors on the resolutions of each Du Val singularity, and compare them with the corresponding Dynkin diagrams.

14. Show that the blow up of a scheme X along an effective Cartier divisor $D \subset X$ is an isomorphism. Show by example that if D is not Cartier then the blow up need not be an isomorphism.