3-MANIFOLDS (EXERCISE SHEET)

Exercise 1 (van Kampen-based warm-up). *(1) Show that the fundamental group of the genus 2 surface is isomorphic to the following*

$$\langle a_1, b_1, a_2, b_2 \mid [a_1, b_1] \cdot [a_2, b_2] = 1 \rangle$$

where $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$.

(2) Show that if M_1 and M_2 are both manifolds of dimension $d \geq 3$ then

$$\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$$

where $G * H$ denotes the free product of the two groups G and H.

Exercise 2. *Show that, for any $n \in \mathbb{N}$, there exists a compact 3-manifold whose fundamental group is isomorphic to the free group generated by n elements.*

Exercise 3. *Show that $S^2 \times \mathbb{R}$ cannot be written as a non-trivial connected sum of two other 3-manifolds.*

Exercise* 4. *Let $A \in \mathrm{SL}(2, \mathbb{Z})$, and let ψ_A be the homeomorphism of the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ induced by A.*

(1) Compute the homology group $\mathrm{H}_1(M, \mathbb{Z})$ of M the suspension of $\mathbb{T}^2$ by ψ_A, depending on the value of A.

(2) Give a presentation of its fundamental group.

Exercise 5 (Poincaré sphere). *By definition, the Poincaré sphere $\mathcal{SP}$ is the quotient of a regular dodecahedron where opposite faces (which are pentagons) have been identified via a positive twist of angle $\frac{\pi}{5}$.*

(1) Show that $\mathcal{SP}$ is a 3-manifold.

(2) Compute its homology groups.

Exercise 6. *Show that $\mathbb{R}^3 \setminus S$ where S is a finite set of cardinality ≥ 2 cannot be the universal cover of a compact manifold.*

Exercise 7. *Let* $\mathcal{H}(\mathbb{R}) = \{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{R}\}$ *and* $\mathcal{H}(\mathbb{Z}) = \{ \begin{pmatrix} 1 & n & m \\ 0 & 1 & l \\ 0 & 0 & 1 \end{pmatrix} \mid n, m, l \in \mathbb{Z}\}$. *Show that $\mathcal{H}(\mathbb{R})/\mathcal{H}(\mathbb{Z})$ is homeomorphic to the mapping torus of $\mathbb{T}^2$ by the homeomorphism induced by* $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Exercise* 8. ***(Warning: you might have to look up a bit of hyperbolic geometry to do this exercise, although nothing fancy, just basic definitions)***

(1) Show that the quotient of $\mathbb{H}^3$ the hyperbolic 3-space by $\mathrm{PSL}(2, \mathbb{Z}[i])$ is neither compact nor a manifold.

(2) Show that there is a finite index subgroupΓ of $\mathrm{PSL}(2, \mathbb{Z}[i])$ such that $\mathbb{H}^3/\Gamma$ is a manifold.